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Lebesgue functions for polynomial interpolation – a survey

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Dedicated to Ted Rivlin, on the occasion of his 70th birthday

Thirty years ago in a paper [44] Luttmann and Rivlin demonstrated the importance of the study of the Lebesgue functions for polynomial interpolation through investigation of their local maxima. Based on extensive numerical calculations they formulated several conjectures which stimulated a great deal of research. In the present paper we give a review of the some results obtained along this line.

Keywords: Lagrange interpolation, Lebesgue functions, Lebesgue constants.

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1. Introduction: Notations and elementary properties of the Lebesgue function

Let $X = \{x_k = x_{kn} : k = 0, 1, \ldots, n; \quad n = 1, 2, 3, \ldots\}$ be a triangular array of nodes such that for each n

$$-1 \le x_0 < x_1 < \ldots < x_n \le 1,$$

and denote by $C[-1, 1]$ the Banach space of continuous functions on $[-1, 1]$ equipped with the uniform norm

$$\|f\| = \max_{x \in [-1,1]} |f(x)|.$$

To any $f \in C[-1, 1]$ there corresponds a unique interpolation polynomial $L_n(f; X; x)$ of degree at most n coinciding with $f(x)$ at the nodes of the nth row of X, which can be expressed in the Lagrangian form:

$$L_n(f; X; x) = \sum_{k=0}^{n} f(x_k) l_k(X; x),$$

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where

$$l_k(X;x) = l_{kn}(X;x) = \frac{\omega(x)}{(x-x_k)\omega'(x_k)}, \qquad k = 0,1,2,\ldots,n,$$

and

$$\omega(x) = (x-x_0)(x-x_1)\ldots(x-x_n).$$

In the study of convergence properties of the Lagrange interpolation polynomials, the behavior of the Lebesgue functions

$$\lambda_n(X;x) = \sum_{k=0}^{n} |l_k(X;x)| \tag{1}$$

and the Lebesgue constants

$$\Lambda_n(X) = \max_{x\in[-1,1]} \lambda_n(X;x) \tag{2}$$

is of fundamental importance (see e.g. [18,25,53,64],). Note, in particular, that the operator norm of $L_n(X;.)$ (as an operator on $C[-1,1]$) equals the supnorm of its Lebesgue function:

$$\|L_n(X)\| = \Lambda_n(X) = \max_{-1\le x\le 1} \lambda_n(X;x).$$

We start with the formulation of elementary properties of $\lambda_n(X;x)$ (see e.g. [44]).

(i) For $n \ge 2$ the function $\lambda_n(X;x)$ is a piecewise polynomial satisfying $\lambda_n(X;x) \ge 1$ with equality only at the nodes x_k $(k = 0,1,\ldots,n)$.

(ii) $\lambda_n(X;x)$ has precisely one local maximum on $(x_{k-1},x_k), k = 1,2,\ldots,n$, which will be denoted by $\mu_k(X)$.

(iii) $\lambda_n(X;x)$ monotone decreasing and convex in $(-1,x_0)$ and monotone increasing and convex in $(x_n,1)$. If $-1 < x_0$ and $x_n < 1$ we will denote $\lambda_n(X;-1)$ and $\lambda_n(X;1)$ by $\mu_0(X)$ and $\mu_{n+1}(X)$ correspondingly. Of particular importance are the largest and the smallest local maxima of $\lambda_n(X;x)$ in $[-1,1]$ which will be denoted correspondingly by $M_n(X)$ and $m_n(X)$. Finally let $\delta_n(X) = M_n(X) - m_n(X)$.

(iv) The equality $\lambda_n(X;x) = \lambda_n(X;-x), \quad x \in [-1,1]$ holds if and only if $x_{n-k} = -x_k, \quad k = 0,1,\ldots,n.$

(v) Let $Z = \{z_k\}_{k=0}^n$ and $Y = \{y_k\}_{k=0}^n$ be two sets of interpolation nodes related by an affine transformation $y_k = \alpha z_k + \beta$. Then

$$\lambda_n(Z;z) = \lambda_n(Y,\alpha z + \beta).$$

Remark 1

In [51] Neuman claimed that the following additional property of the Lebesgue function holds: for any matrix X in each of the intervals (x_{k-1},x_k), $k = 1,2,\ldots,n$ the function $\lambda_n(X;x)$ is strictly concave. Unfortunately, this claim is false, as the following example illustrates: Let $X = \{-1,-a,0,a,1\}$ and suppose

that $a < 0.45$. Then one can verify that in the interval $(a, 1)$ the second derivative $\lambda_4''(X; x)$ has positive value when x approaches a.

2. The behavior of the Lebesgue function for specific sets of interpolation points

For some sets of nodes which are of special importance in the interpolation theory, such as equidistant nodes, Chebyshev roots and extrema and others, the behavior of the Lebesgue function is well investigated. We start our discussion with the case of equally spaced points.

2.1. *Equidistant nodes*

Although it was demonstrated by Runge at the beginning of the century [57] that the set of equally spaced points $E = \{x_k = -1 + 2k/n, \quad k = 0, 1, \ldots, n\}$ is a bad choice for Lagrange interpolation, there was a lot of activity in studying the behavior of the Lebesgue function corresponding to equidistant nodes.

As was pointed out by Rivlin in [56], the starting point of this research goes back to Heinrich Tietze who in 1917 published a paper [66] devoted to the behavior of the relative maxima of the Lebesgue function $\lambda_n(E; x)$. Tietze proved that these maxima are strictly decreasing from the outside towards the middle of the interval. For the largest local maximum $M_n(E) = \Lambda_n(E)$ Tietze obtained a rather conservative lower bound. As to the smallest one $m_n(E)$, Tietze observed that

$$\lambda_{2r+1}(E; 0) = m_{2r+1}(E) = m_{2r}(E) = \lambda_{2r}(E; \pm 1/2r) \tag{3}$$

and pointed out that $m_{2r}(E)$ *coincides* with the following quantity

$$G_r = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \ldots + \left(\frac{1 \cdot 3 \cdots (2r-1)}{2 \cdot 4 \cdots 2r}\right)^2, \tag{4}$$

which appears in the famous paper of Landau of 1913 [42] and which is known in the literature as the Landau constant (in the modern terminology G_n is the norm of the Taylor projection in the space of functions analytic inside the unit circle and continuous on the boundary). As was shown by Landau himself

$$G_n \sim \frac{1}{\pi} \log n, \qquad n \to \infty. \tag{5}$$

The paper of Tietze remained unnoticed.

A similar fate awaited the paper of Turetskii [68] published in Russian in 1940 in a local mathematical journal which was unavailable to the western mathematicians. (The result was reproduced in his book of 1968 [69] which is also published in Russian.) In this paper Turetskii rediscovered the monotone behavior of the local maxima and found the following asymptotic expression for the largest

maximum

$$\Lambda_n(E) \sim \frac{2^{n+1}}{en\log n}, \qquad n \to \infty. \tag{6}$$

He also studied the behavior of $m_n(E)$ and showed that it tends asymptotically to $\log n/\pi$.

The third paper with similar results was published in 1961 by Schönhage [58], who was unaware of the results of Tietze and Turetskii. Schönhage restricted himself to the localization of the greatest and the smallest relative maxima. Further he found a little bit more precise than (6) asymptotic expression for $\Lambda_n(E)$, namely

$$\Lambda_n(E) \sim \frac{2^{n+1}}{en(\log n + \gamma)}, \qquad n \to \infty, \tag{7}$$

where $\gamma = 0.577\ldots$ is Euler's constant. As to the least local maxima Schönhage obtained the following rather conservative estimate:

$$m_n(E) < \frac{2}{\pi}[\log(n+2) + \log 2 + \gamma]. \tag{8}$$

Most of the recent results dealing with equidistant points have been devoted to refinement of the above estimates. Thus Mills and Smith in [48] improved (7) by finding an asymptotic expansion for $\Lambda_n(E)$ of the following form:

$$\log \Lambda_n(E) = (n+1)\log 2 - \log n - \log\log n - 1 + \sum_{k=1}^{m} \frac{A_k}{(\log n)^k} + O\left(\frac{1}{(\log n)^{m+1}}\right), \qquad m = 1, 2, 3, \ldots, \tag{9}$$

where

$$A_1 = -\gamma, \qquad A_2 = \gamma^2/2 - \pi^2/12, \qquad A_3 = -\gamma^3/3 + \gamma\pi^2/6 - \zeta(3)/3, \ldots,$$

and where $\zeta(3) = \sum_{r=1}^{\infty} r^{-3}$. Note that it appears to be a difficult problem to obtain an explicit general formula for the coefficients A_k.

Note also that Trefethen and Weideman [67] suggested an elegant proof of the two-sided estimate for the Lebesgue constants $\Lambda_n(E)$

$$\frac{2^{n-2}}{n^2} < \Lambda_n(E) < \frac{2^{n+3}}{n}, \qquad n \geq 1,$$

and made some references to additional partial results concerning the rate of growth of $\Lambda_n(E)$.

As to the smallest local maximum $m_n(E)$, it should be pointed out that since its relationship to the Landau constant was ignored, the results concerning the behavior of $m_n(E)$ duplicate the corresponding results for G_n. We would like to mention the result of Men'shikov [47] and a recent paper of Mills and Smith [49], in which the monotonicity of local maxima of $\lambda_n(E; x)$ was rediscovered and the following

expression for $m_n(E)$ was obtained

$$m_n(E) = 1 + \sum_{k=1}^{n} \left[\frac{(1/2)_k}{(1)_k} \right], \qquad n = 2r + 1,$$

where $(a)_k = a(a+1)(a+2)\dots(a+k-1)$. Unfortunately Mills and Smith did not identify this expression as the Landau constant. As a result they developed an asymptotic expansion for $m_n(E)$ and obtained estimates for finite n which are similar (and a little bit less precise) than the corresponding results for G_n obtained earlier by Brutman [13], Wilkins [71] and Falaleev [27].

2.2. *Chebyshev nodes*

For the Chebyshev nodes $T = \{x_k = -\cos[(2k+1)\pi/(2n+2)], \quad k = 0, 1, \dots, n\}$, consisting of the roots of the Chebyshev polynomial of the first kind the behavior of $\lambda_n(T; x)$ has been thoroughly investigated.

An asymptotic value of $\Lambda_n(T)$ was established by Bernstein in [3]

$$\Lambda_n(T) \sim \frac{2}{\pi} \log(n+1), \qquad \text{as} \quad n \to \infty. \tag{10}$$

Natanson in his monograph [50] obtained the following estimate valid for all finite n:

$$\Lambda_n(T) < 8 + \frac{4}{\pi} \log(n+1).$$

This estimate was improved by Berman [2]

$$\Lambda_n(T) < 4\sqrt{2} + \frac{2}{\pi} \log(n+1)$$

and later by Ivanov and Zadiraka [35]

$$\Lambda_n(T) < 2 + \frac{2}{\pi} \log(n+1).$$

Further improvement was derived from the behavior of the Lebesgue function. Luttman and Rivlin [44], in view of extensive numerical computations, conjectured that

$$\Lambda_n(T) = \lambda_n(T; 1), \tag{11}$$

and using the representation

$$\lambda_n(T; 1) = \frac{1}{n+1} \sum_{k=0}^{n} \cot \frac{2k+1}{4(n+1)} \pi, \tag{12}$$

established the following asymptotic formula

$$\lim_{n \to \infty} \left[\lambda_n(T; 1) - \frac{2}{\pi} \log(n+1) \right] = \frac{2}{\pi} \left(\gamma + \log \frac{8}{\pi} \right) := a_0 = 0.9625\dots . \tag{13}$$

Later (11) was proved by Ehlich and Zeller [20] (see also Powell [52]). Rivlin in [54] showed that the asymptotic tendency in (13) is of monotone character and obtained the following sharp estimate:

$$a_0 + \frac{2}{\pi}\log(n+1) < \Lambda_n(T) \leq 1 + \frac{2}{\pi}\log(n+1), \qquad n = 0, 1, 2, \ldots . \tag{14}$$

In addition to this estimate which is valid for all finite n, the following asymptotic expansion, which generalizes (13), was established independently by Shivakumar and Wong [59], Dzyadyk and Ivanov [19] and Günttner [30]:

$$\Lambda_n(T) \sim \frac{2}{\pi}\log(n+1) + a_0 + \sum_{k=1}^{\infty} \frac{a_k}{(n+1)^{2k}}, \qquad \text{as } n \to \infty, \tag{15}$$

where

$$a_k = \frac{(-1)^{k+1}(2^{2k-1}-1)^2 \pi^{2k-1} B_{2k}^2}{4^{k-1} k(2k)!}, \qquad k = 1, 2, \ldots,$$

and B_k are the Bernoulli numbers. Further, the error has the same sign as, and is in absolute value less than, the first term neglected.

The study of the Lebesgue function $\lambda_n(T; x)$ via its local maxima was initiated by Luttmann and Rivlin in [44], where on the basis of numerical computations they conjectured that the relative maxima of $\lambda_n(T; x)$ are strictly decreasing as we proceed into the interval $[-1, 1]$ from the endpoints through its midpoint. This conjecture was proved by Brutman [11] (see also [30]) who also found the following estimate for the smallest local maximum

$$m_n(T) > \frac{2}{\pi}\log(n+1) + \chi, \tag{16}$$

where

$$\chi = \frac{2}{\pi}\left(\gamma + \log\frac{4}{\pi}\right) = 0.521251 \ldots . \tag{17}$$

Inequality (16), together with (14), implies that the deviation between any two local maxima of $\lambda_n(T; x)$ does not exceed 0.4788. Günttner [30] proved the following numerical improvement of this result

$$M_n(T) - m_n(T) \leq \frac{2}{\pi}\log 2 = 0.44127 \ldots .$$

The same bound was found later by Dzyadyk and Ivanov [19], who had no knowledge of the above-mentioned papers of Brutman and Günttner.

Very recently [32] the estimate (16) was improved as follows

$$m_n(T) > \frac{2}{\pi}\log(n+1) + \chi + \frac{\pi}{18(n+1)^2} - \frac{49\pi^3}{10800(n+1)^4}. \tag{18}$$

Finally, it was shown by Brutman in [11] that the case of the T-nodes is a special one in the sense that for other sets of nodes which are of interest in the theory of

interpolation, such as the Chebyshev extrema, the roots of the Chebyshev polynomials of the second type and others, the largest and the smallest relative maxima are of different orders of magnitude.

2.3. *Extended Chebyshev nodes*

The extended Chebyshev nodes is the set $\hat{T}$ obtained from the T-set by linear transformation which maps $[x_0, x_n]$ into $[-1, 1]$, namely

$$\hat{T} = \{x_k = -\cos[(2k+1)\pi/(2n+2)]/\cos[\pi/(2n+2)], \qquad k = 0, 1, \ldots, n\}.$$

It was demonstrated in [44] and [11] that the main decrease in the values of the local maxima of $\lambda_n(T; x)$ takes place between the value at 1 and the next local maximum. Therefore the Lebesgue constant $\Lambda_n(\hat{T})$ is essentially less than $\Lambda_n(T)$. Brutman [11] found the following estimate valid for all finite $n \geq 1$

$$\Lambda_n(\hat{T}) < 0.7219 + \frac{2}{\pi}\log(n+1), \tag{19}$$

and thus established that

$$\delta_n(\hat{T}) = M_n(\hat{T}) - m_n(\hat{T}) < 0.201, \qquad n = 1, 2, \ldots . \tag{20}$$

By further analysis Günttner [30] showed that

$$\delta_n(\hat{T}) \leq 0.0196, \qquad n \geq 70, \tag{21}$$

and also observed, referring to [46], that asymptotically

$$\lim_{n\to\infty} \delta_n(\hat{T}) = \frac{2}{\pi}\log 2 - \frac{4}{3\pi} = 0.01686.... \tag{22}$$

Finally, for the Lebesgue constants $\Lambda_n(\hat{T})$ Günttner [31] proved the existence of the asymptotic expansion of the following form:

$$\Lambda_{n-1}(\hat{T}) = \frac{2}{\pi}\log n + b_0 + \frac{b_1}{\log n} + \ldots + \frac{b_k}{(\log n)^k} + R_n^{(k)}, \qquad k = 0, 1, 2, \ldots, \tag{23}$$

where

$$b_0 = a_0 - 4/(3\pi) = 0.5381\ldots, \qquad b_1 = 16/(81\pi^3) = 0.006371\ldots, \qquad \ldots$$

and

$$R_n^{(k)} = O\left(\frac{1}{(\log n)^{k+1}}\right).$$

It should be noted that, in contrast to the expansion (15) for the Chebyshev nodes, an explicit general formula for the coefficients b_i in (23) is unknown.

2.4. *Chebyshev extrema*

The interpolation at the Chebyshev extrema $U = \{x_k = -\cos(k\pi/n), \quad k = 0,$

$1, \ldots, n\}$ was studied first by Ehlich and Zeller [20] and McCabe and Phillips [46]. It was found that

$$\Lambda_n(U) = \begin{cases} \Lambda_{n-1}(T), & n = 1, 3, \ldots, \\ \Lambda_{n-1}(T) - \alpha_n, \ 0 \le \alpha_n < 1/n^2, & n = 2, 4, 6, \ldots . \end{cases} \tag{24}$$

This estimate has appeared as well in Rivlin [54], where interest was expressed in "closer scrutiny of the quantities α_n". By estimating the degree of asymmetry of $\lambda_n(U; x)$ in the interval of interest Brutman [14] improved (24) as follows

$$\frac{\pi/8}{(2n)^2} < \alpha_{2n} < \frac{2\sqrt{2} - 2}{(2n)^2}.$$

He also studied the behavior of the Lebesgue function $\lambda_n(U; x)$ in $[-1, 1]$ and proved that the corresponding relative maxima are monotone increasing from the outside towards the middle of the interval. Moreover, it was shown that, in contrast to the case of the Chebyshev roots, the sequence $\{m_n(U)\}_{n=1}^{\infty}$ is bounded.

2.5. *Some other nodes*

In those cases when there is no explicit expression for the interpolation nodes the results are usually restricted to the study of the order of the Lebesgue constants.

Thus the order of the Lebesgue constants corresponding to the roots of Jacobi polynomials was found by Szegö in his classical monograph [65]. Namely, he showed that if J is made up of the zeros of the Jacobi polynomials with parameters (α, β), $\alpha, \beta > -1$ and $\gamma = \max(\alpha, \beta)$, then

$$\Lambda_n(J) = \begin{cases} O(n^{\gamma+1/2}), & \gamma > -1/2, \\ O(\log n), & \gamma \le -1/2. \end{cases} \tag{25}$$

Of special interest is the case of Fekete nodes, namely, the matrix F of nodes, the nth row of which consists of the zeros of $(x^2 - 1)P_n'(x)$, where $P_n(x)$ is the Legendre polynomial of degree n. As was proved by Fejér [28] these nodes are the solution to the following optimal problem: find a set of nodes minimizing

$$\Phi_n(X) = \max_{-1 \le x \le 1} \sum_{k=0}^{n} [l_k(X; x)]^2.$$

Fejér established that $\Phi_n(F) = 1$, which, in conjunction with Schwarz' inequality gives

$$\Lambda_n(F) \le (n+1)^{1/2}.$$

Luttmann and Rivlin [44] performed numerical computations of $\Lambda_n(F)$ for $n = 3, \ldots, 40$ and on this basis conjectured that

(a) $\Lambda_n(F) < \Lambda_n(T), \ n > 3.$

(b) The relative maxima of $\lambda_n(F;x)$ on subintervals between nodes decrease in $[0,1]$ and therefore $\lambda_n(F;x)$ attains its maximum on $[-1,1]$ at $x=0$ for even n and "near" $x=0$ for odd n.

Although both conjectures are still unsettled, Sündermann [61] succeeded in proving that

$$\Lambda_n(F) = O(\log n). \tag{26}$$

It is well known that the U-nodes coincide with the extreme points in best uniform approximation of x^{n+1} by polynomials of degree $\leq n$. Generalizing this approach, Henry and Swetits [33,34] considered interpolation at the nodes induced by extreme points of the error function of the best polynomial approximation for some other functions. In particular, they proved in [33] that when the approximated function is $x^{n+1} + \sigma x^n$, $0 \leq \sigma \leq (n+1)\tan^2((\pi/(2n+2))$, namely when the interpolation nodes coincide with the extreme points of the appropriate Zolotorev polynomials, the corresponding Lebesgue constants are of precise order $O(n)$. On the other hand, it is shown in [34] that for some approximating functions, the corresponding Lebesgue constants grow logarithmically.

Finally we would like to mention a recent paper of Mason and Elliot [45] in which they considered the *weighted* interpolation based on the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ corresponding to the quartet of parameters $\alpha = \pm 1/2, \beta = \pm 1/2$. (Note that the Jacobi polynomials corresponding to $(\alpha,\beta) = (1/2,-1/2)$ and $(\alpha,\beta) = (-1/2,1/2)$ are usually designated in the literature as the Chebyshev polynomials of the third and fourth kind respectively.) The behavior of the appropriately weighted Lebesgue functions is treated there and the norms of the resulting projections are computed numerically. Based on these computations several conjectures are formulated and some asymptotic formulas for the operator norms are given.

3. Optimal choice of interpolation nodes

Since $L_n(X)$ is a linear projection from $C[-1,1]$ onto the subspace π_n of polynomials of degree $\leq n$, and therefore the efficiency of approximation by $L_n(X)$ is governed by

$$\| f - L_n(f,X)\| \leq [1 + \Lambda_n(X)]dist(f,\pi_n),$$

it is desirable to minimize $\Lambda_n(X)$. As well-known (see e.g. [53, p.100]) there exists an optimal set of nodes, namely a set $X^\star$ of $(n+1)$ distinct points in $[-1,1]$ such that

$$\Lambda_n^\star := \Lambda_n(X^\star) = \min_X \Lambda_n(X).$$

The characterization of the optimal configurations of nodes $X^\star$ and the estimation of the minimal Lebesgue constants $\Lambda_n^\star$ are among the most intriguing problems of interpolation theory.

The fact that Λ_n^* is of logarithmic order of growth and that this order is attainable for the Chebyshev nodes was found very early by Faber [26] and Bernstein [3]. In 1931 Bernstein [4] returned to the problem of estimating the optimal Lebesgue constants and found that asymptotically

$$\Lambda_n^* \sim \frac{2}{\pi}\log(n+1), \qquad n \to \infty.$$

In 1961 Erdös [23] proved that there are positive constants C_1 and C_2 such that

$$\frac{2}{\pi}\log(n+1) - C_1 \leq \Lambda_n^* \leq \frac{2}{\pi}\log(n+1) + C_2, \qquad n = 1, 2, \ldots$$

The problem of characterizing an optimal set X^* was posed by Bernstein in the above-mentioned paper of 1931 in which he wrote: "It seems plausible that the greatest of the relative maxima of $\lambda_n(X;x)$ will be minimized if all these maxima will be equal. But I was able to prove this statement only under the assumption that n grows infinitely ..."

To formulate this conjecture more precisely, we restrict ourselves to the *canonical* sets of nodes, which include the end-points as the interpolation nodes. Let n be fixed and let $X = X_{n+1} = \{-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1\}$ denote an arbitrary canonical set of interpolation nodes. Let $\mu_j(X), j = 1, 2, \ldots, n$, as before, denote the local maxima of the Lebesgue function $\lambda_n(X;x)$. Then the conjecture of Bernstein claims that if

$$\mu_1(X^*) = \mu_2(X^*) = \ldots = \mu_n(X^*), \tag{27}$$

then X^* is an optimal set of nodes .

Erdös [21,22] amplified Bernstein's conjecture as follows: There is a unique *canonical* set X^* for which (27) holds, and for any set X

$$m_n(X) = \min_i \mu_i(X) \leq \Lambda_n(X^*) \leq \max_i \mu_i(X) = M_n(X). \tag{28}$$

The conjectures of Bernstein and Erdös were proved by Kilgore [37] and de Boor and Pinkus [6].

In spite of this nice characterization, the optimal nodes as well as the optimal Lebesgue constants are not known explicitly. The first result towards estimating the optimal norm which makes use of the Erdös inequality (28) is due to Brutman [11], who proved (see estimates (16) and (19) of the previous section) that the set of the extended Chebyshev nodes $\hat{T}$ provides a very good approximation to the optimal set in the sense that for all $n \geq 1$ the following inequality holds:

$$\frac{1}{2} + \frac{2}{\pi}\log(n+1) < m_n(\hat{T}) < \Lambda_n(X^*) < M_n(\hat{T}) < \frac{3}{4} + \frac{2}{\pi}\log(n+1). \tag{29}$$

For sufficiently large n this estimate was sharpened by Günttner (see (21), (22)).

At this point it is worthwhile to mention that describing these results in the second edition of his monograph "Chebyshev polynomials" Rivlin concludes: "The readily available extended Chebyshev array $\hat{T}$ is, *for all practical purposes*

as useful as the optimal nodes." However, as we all know, mathematical curiosity goes far beyond "all practical purposes", and the problem of analytical description of the optimal matrix of nodes is considered by pure mathematicians as a great challenge.

A related, but less hopeless problem is to find a sharp estimate for the optimal Lebesgue constants. The first step in this direction was made by Vértesi [70] who found the order of the optimal Lebesgue constants by constructing the following modification of the Chebyshev matrix. Let $\psi_n = a/(n \log n)$ with an appropriate constant $a > 0$ and let

$$T_{sh} = \{y_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, y_n\}, \qquad n = 2, 3, \ldots,$$

where $x_k = -\cos\theta_k = -\cos[(2k-1)\pi/(2n)]$ are the Chebyshev roots, $1 \le k \le n$, and

$$y_1 = -y_n = -\cos(\theta_1 + \psi_n).$$

Thus T_{sh} is a set of nodes obtained from T by an appropriate shift of the first and the last nodes in the direction of the origin. After shifting the standard extension of the nodes is performed (namely the linear transformation which maps the first and the last nodes to -1 and 1 correspondingly.) The resulting matrix will be denoted by $\hat{T}_{sh}$ and will be called the shifted-extended Chebyshev matrix.

It was proved in [70] that for $\hat{T}_{sh}$ corresponding to $a = \pi/8$ the following asymptotic estimate holds:

$$\Lambda_{n-1}(\hat{T}_{sh}) = \frac{2}{\pi}\log n + \chi + O\left(\left(\frac{\log\log n}{\log n}\right)^2\right), \tag{30}$$

where χ was defined by (17). An application of the Erdös inequality (28) in the form

$$m_{n-1}(T) = m_{n-1}(\hat{T}) \le \Lambda^*_{n-1} \le M_{n-1}(\hat{T}_{sh}),$$

along with the inequalities (18) and (30) leads to the following estimate

$$\frac{\pi}{18n^2} + O\left(\frac{1}{n^4}\right) < \Lambda^*_{n-1} - \frac{2}{\pi}\log n - \chi < const\left(\left(\frac{\log\log n}{\log n}\right)^2\right). \tag{31}$$

From (31) it follows that the norm of the interpolating operator corresponding to the shifted-extended Chebyshev matrix $\hat{T}_{sh}$ gives *asymptotically* the order of the optimal norm.

But we have also performed the numerical computations of $\Lambda_n(\hat{T}_{sh})$ for the following values of n: $n = 3 \div 10, 20, 30, \ldots, 100, 150, 200$. The computed results are presented in table 1, along with the corresponding values of the Lebesgue constants for the extended Chebyshev nodes and the values of the optimal Lebesgue constants (which have been found numerically in [1] and presented here with a kind permission of the authors.)

These numerical results show that even for finite values of n (starting with $n = 5$)

Table 1
The values of the Lebesgue constants for the optimal X^*, extended Chebyshev $\hat{T}$ and shifted-extended Chebyshev $\hat{T}_{sh}$ nodes.

n	$\Lambda_n(X^*)$	$\Lambda_n(\hat{T}_{sh})$	$\Lambda_n(\hat{T})$
3	1.422 920	1.448 083	1.429 873
4	1.559 490	1.575 680	1.570 167
5	1.672 210	1.683 646	1.685 140
6	1.768 135	1.776 834	1.782 530
7	1.851 599	1.858 521	1.866 999
8	1.925 458	1.931 112	1.941 573
9	1.991 685	1.996 560	2.008 327
10	2.051 706	2.056 087	2.068 744
20	2.460 788	2.463 129	2.479 193
30	2.708 082	2.709 645	2.726 693
40	2.885 809	2.887 067	2.904 441
50	3.024 619	3.025 651	3.043 229
60	3.138 527	3.139 389	3.157 102
70	3.235 120	3.235 887	3.253 659
80	3.318 973	3.319 660	3.337 477
90	3.393 058	3.393 677	3.411 530
100	3.459 415	3.459 973	3.477 858
150	3.715 393	3.715 787	3.733 720
200	3.897 466	3.897 772	3.915 713

the Lebesgue numbers $\Lambda_n(\hat{T}_{sh})$ are closer to the optimal Lebesgue constants $\Lambda_n(X^*)$ in comparison with the Lebesgue numbers $\Lambda_n(\hat{T})$ corresponding to the extended Chebyshev nodes.

We conclude this section with several remarks:

Remark 2
The conjectures of Bernstein and Erdös characterize optimal interpolation of continuous functions on a finite interval $[a, b]$ by algebraic polynomials. In a series of papers [38–40] Kilgore demonstrated that these conjectures remain valid when polynomials are replaced by some other finite dimensional subspaces of $C[a, b]$. De Boor and Pinkus [6] extended these conjectures for trigonometric interpolation of continuous 2π-periodic functions. It is believed that the Bernstein-Erdös conditions characterize the minimum norm interpolation induced by arbitrary Chebyshev system with constants (see e.g. [36,38,43]).

Remark 3
The numerical results presented above show that the estimate on the right-hand side of (31) is rather conservative, and one can expect that

$$\Lambda_{n-1}^{\star} = \frac{2}{\pi}\log n + \chi + O\left(\frac{1}{n^2}\right). \tag{32}$$

Remark 4
In view of the conjectures of Bernstein and Erdös, it is natural to use the quantity $\delta_n(X) = M_n(X) - m_n(X)$ as the measure of "closeness" of the matrix X to the optimal one. One can verify that

$$\lim_{n\to\infty} \delta_n(\hat{T}_{sh}) = \frac{5}{3\pi} - \frac{2}{\pi}\log 2 = 0.0892\ldots . \tag{33}$$

Comparing this result with (22), we conclude that the extended Chebyshev matrix is superior in comparison with the shifted-extended Chebyshev matrix, as far as closeness to the optimal matrix is concern.

4. Complex case: Optimal interpolation of continuous functions in the unit disc

Let $D = \{z : |z| \le 1\}$ and denote by $C(D)$ the Banach space of continuous functions in D equipped with the uniform norm. Let $Z = Z_{n+1} = \{z_k\}_{k=0}^n$ be a set of $n+1$ distinct points in D and let $\lambda_n(Z;z) = \sum_{k=0}^n |l_k(Z;z)|$ be the corresponding Lebesgue function.

Since $\lambda_n(Z;z)$, being the sum of absolute values of analytic functions, is a subharmonic function of z, it attains its maximal value on $\Gamma = \{z : |z| = 1\}$, and therefore

$$\|L_n(Z)\| = \max_{z\in D} \lambda_n(Z;z) = \max_{0\le\theta\le 2\pi} \lambda_n(Z;e^{i\theta}). \tag{34}$$

The minimal norm interpolation problem is well-studied for the case corresponding to the location of nodes on the boundary Γ of the disc D. It was conjectured by Erdös in [24] and proved by Brutman and Pinkus [12,17] that the optimal choice of interpolation nodes on Γ corresponds to the sets of equally spaced knots $E_\Gamma = \exp(2k\pi i/(n+1) + \alpha)\}_{k=0}^n$, some α.

The proof of Brutman and Pinkus is based on the equivalence between the optimal interpolation problem on the unit circle and the corresponding problem for trigonometric interpolation. (For even number of interpolation points, instead of the classical trigonometric polynomials, the polynomials of half-angles are used.) As a result of this equivalence, a reduction to the real context was possible. As mentioned in [16], it will be interesting to find a direct proof of the Erdös conjecture concerning minimal norm interpolation on the unit circle.

Remark 5
The explicit expression for the optimal norm is due to Gronwall [29]:

$$\|L_n(E_\Gamma)\| := \Lambda_n(E_\Gamma) = \frac{1}{n+1}\sum_{k=0}^{n} \csc\frac{2k+1}{2n+2}\pi, \tag{35}$$

and comparison of (35) with (11), (12) yields

$$\Lambda_n(T) = \Lambda_n(E_\Gamma). \tag{36}$$

As was pointed out in [16], in spite of this identity, curiously, for a long time articles were published concerning the behaviour of these two sequences separately

When the interpolation nodes are not restricted to the boundary of D, the situation becomes more complicated and, to the best of our knowledge, the problem of characterizing optimal interpolation nodes in this general setting is an open one. Based on partial analytic results as well as the results of numerical computations, Brutman [16] suggested that the Erdös conjecture remains valid when the interpolation nodes are arbitrarily chosen points in the unit disc, namely that the following relationship holds:

$$\min_{Z\in D}\max_{z\in\Gamma}\lambda_n(Z;z)=\min_{Z\in\Gamma}\max_{z\in\Gamma}\lambda_n(Z;z). \tag{37}$$

5. Concluding remarks

In the present survey we restricted our discussion to the Lagrange interpolation of univariate functions on the finite interval.

Concerning interpolation on unbounded intervals, we confine ourselves to mentioning two recent papers [41] and [60]. In the first Kilgore proved that the Bernstein and the Erdös conjectures may be extended to the interpolation with Hermite weight on the real line as well as with the Laguerre weight on the half-line. The second paper, due to Sklyarov, deals with a favorable choice of interpolation points in the space $C(-\infty,\infty)$.

The problem of optimal or even near optimal interpolation in the multivariate case is much more complicated. Some results in this directions may be found in the works of Bos [7–10], Bloom [5] and Sündermann [62,63].

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